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MATH 204-Real Analysis II Time: 2 HRS 2021/2022 Session

Instruction: Answer any FOUR (4) questions

1(a) Show that every differentiable function at a point x_o is continuous at x_o.

(b) Is the converse to the result in (a) above true? Show it.

2(a) State and prove Rolle's Theorem.

(b) If f and g are any two functions defined on some neighborhood of c such that $\lim_{x\to c} f(x) = p$ and $\lim_{x\to c} g(x) = q$, then prove that $\lim_{x\to c}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{p}{q}$, $q \neq 0$.

3(a) If a function f defined on $[a,b]$ satisfies the following conditions:

(i) f is continuous on $[a,b]$;
(ii) differentiable on $]a,b[$,

then prove that there exists at least one real number $c \in]a,b[$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}, \qquad b \neq a$$

(b) If a function f is: (i) continuous on $[a,b]$, differentiable on $]a,b[$ and $f'(x) < 0$ for all $x \in]a,b[$, then show that f is a strictly decreasing function.

4(a) Let $f: D(f) \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $c \in D(f)$. Then, define the following concepts:

3(i) right-continuity of f at c.
3(ii) left-continuity of f at c.

(b) Let the function $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2, & \text{if } x \geq 5 \\ x-1 & \text{if } x < 5 \end{cases}$.

Prove that f is continuous to the right at $x = 5$.

5(a) Verify the Rolle's Theorem for the function $f(x) = x^3 - 4x$ on the interval [-2, 2]. Hence, find the value of c in (-2, 2) such that $f'(c) = 0$.

(b) A function $f: \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Show that f is differentiable at $x = 0$, but $\lim_{x\to 0} f'(x) \neq f'(0)$.

6(a) Prove that every continuous function on a closed interval is uniformly continuous.

(b) Show whether the function $f(x) = x^2$ is uniformly continuous on [-1, 1].

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(1a) Proof: By Contradiction.

Suppose f is a differentiable function at $x = x_0$ then given $\varepsilon > 0$ $\exists$ $\delta > 0$ $\ni$
$\left|\frac{f(x)-f(x_0)}{x-x_0} - f'(x)\right| < \varepsilon$ $\forall$ $x \in D(f)$ satisfying $|x - x_0| < \delta$.

Suppose f is not continuous for $|x-x_0| < \delta$ then $|f(x) - f(x_0)| > \varepsilon$

$\Rightarrow \left|\frac{f(x)-f(x_0)}{x-x_0}\right| > \frac{\varepsilon}{|x-x_0|} \xrightarrow[x \to x_0]{} ?$

So f is not differentiable at $x = x_0$ Since $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ does not exist.

Hence, differentiability at $x = x_0$ implying Continuity at $x = x_0$ $\square$.

(1b) The Converse of the statement is not always true, i.e f may be Continuous but not differentiable at a point.

Here, we prove by example
$f(x) = |x|$ at $x = 0$.

To show that f is continuous, Let $\varepsilon > 0$ be given $\exists$ $\delta(\varepsilon) > 0 \ni |x - 0| < \delta \Rightarrow |x| < \delta$

$|f(x) - f(0)| = |x| - |0| = |x| < \delta = \varepsilon$

$\Rightarrow f$ is Continous.

Also for differentiability at $x = 0$

$f'(0) = \frac{f(x)-f(0)}{x-0} = \frac{|x|-|0|}{x-0} = \frac{|x|-0}{x-0} = \frac{|x|}{x}$

Taking the $\lim_{x \to 0} \frac{|x|}{x}$ = Undefine.

Hence f is not differentiable at $x = 0$. $\square$

(2a) Rolle's Theorem: Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ such that $f(a) = f(b)$ and f is differentiable at every point (a,b) then $\exists$ a point $c \in (a,b)$ such that $f'(c) = 0$.

Proof:

if f is continous on $[a,b]$ then $f'(c) = 0$ $\forall$ $c \in (a,b)$. If f is not a Constant on $[a,b]$ then there is either a number $c \in (a,b)$ such that $f(a) > f(b)$ or a number $t \in (a,b)$ such that $f(t) < f(a)$ but $f(a) = f(b)$.

Suppose $f(s) > f(a)$ for $s \in (a,b)$ then $f([a,b]) = [\inf f(x), \sup f(x)]$ $s \in (a,b)$ so that $\sup_{x \in [a,b]} f(x) \geq f(s) > f(x)$.

Since f is Continuous on $[a,b]$ then there exist a number $c \in [a,b]$ such that $f(c) = \sup_{x \in [a,b]} f(x)$.

Hence $f(c) > f(a) = f(b)$ Since $f(c) \neq f(a) = f(b)$.

Then c must lie in the open interval (a,b) i.e $f(c) \geq f(x)$ $\forall$ $x \in [a,b]$

But f is differentiable at $c \Rightarrow f'(c) = 0$.

(2b) Proof: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{p}{q}$

$\lim_{x \to c} f(x) = p$, $\lim_{x \to c} g(x) = q \neq 0$

For every $\varepsilon > 0$, $\exists\ \delta_1 > 0$ such that if $0 < |x-c| < \delta_1$ then
$|f(x) - p| < \frac{\varepsilon |q|}{2}$.

Similarly, there exist $\delta_2 > 0 \ni 0 < |x-c| < \delta_2$ then
$|g(x) - q| < \min\left(\frac{|q|}{2}, \frac{\varepsilon |q|^2}{2(|p|+1)}\right)$.

So, $\frac{f(x)}{g(x)} - \frac{p}{q} = \frac{f(x)q - pg(x)}{g(x)q} = \frac{q(f(x)-p) + p(q-g(x))}{g(x)q}$

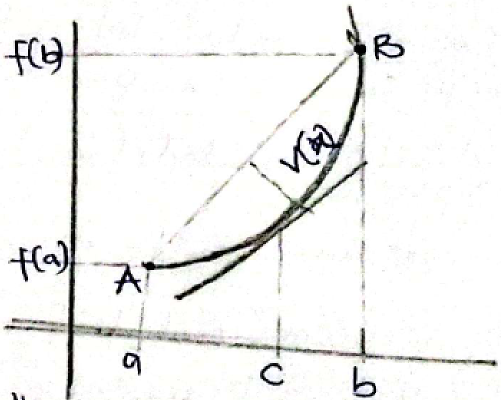
$\Rightarrow \left|\frac{f(x)}{g(x)} - \frac{p}{q}\right| \leq \frac{|f(x)-p|}{|g(x)|} + \frac{|p||g(x)-q|}{|g(x)||q|}$

Since $\lim_{x \to c} g(x) = q \neq 0$ we have $|g(x)| > \frac{|q|}{2}$.

Thus, $\left|\frac{f(x)}{g(x)} - \frac{p}{q}\right| \leq \frac{2}{|q|}|f(x) - p| + \frac{2|p|}{|q|^2}|g(x) - q| = \frac{\varepsilon}{2} + \varepsilon/2 < \varepsilon$

Hence, $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{p}{q}$ $\square$.

(3a) Proof: from the diagram below,



$y - y_1 = m(x - x_1)$

The equation of $AB \Rightarrow y - f(a) = m(x-a)$
$y = f(a) + m(x-a)$

The vertical distance between line A and the area is $v(x) = f(x) - y$
$v(x) = f(x) - f(a) - m(x-a)$ at point $a \Rightarrow b$
$v(a) = v(b)$.

By Rolle's theorem, there exist $c \in (a,b)$ such that $v'(c) = 0$
$v'(x) = f'(x) - m \Rightarrow v'(c) = f'(c) - m = 0$
$\Rightarrow f'(c) = m$.
$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$ $\square$.

(3b) If a function f is (i) continuous on $[a,b]$, differentiable on $]a,b[$ and $f'(x)<0$ $\forall$ $x\in]a,b[$ then we show that f is strictly decreasing function

Let $x_1, x_2 \in [a,b]$ with $x_1 < x_2$ then by MVT,

$\exists\, c\in(x_1,x_2) \ni f(x_2)-f(x_1) = f'(c)(x_2-x_1)$.

Since $x_2 - x_1 > 0$ and $f'(c)<0$

$\Rightarrow f(x_2)-f(x_1)<0 \Rightarrow f(x_2)<f(x_1)$

Thus, f is strictly decreasing on $[a,b]$.

(4a) i) Right-Continuity of f at c: if $\lim_{x\to c^+} f(x) = f(c)$

A function is said to be continuous at the right if $x=c$ if given $\varepsilon>0$ $\exists$ $\delta=\delta(c) \ni |f(x)-f(c)|<\varepsilon$ $\forall x\in D(f)$ satisfying $c<x<c+\delta$.

ii) Left-Continuity of f at c if $\lim_{x\to c^-} f(x)=f(c)$

A function is said to be continuous at the left of $x=c$ if given $\varepsilon>0$ $\exists$ $\delta=\delta(c)\ni |f(x)-f(c)|<\varepsilon$ $\forall x\in D(f)$ satisfying $c-\delta<x<c$.

(4b) $f(x) = \begin{cases} x^2, & \text{if } x\geq 5 \\ x-1, & \text{if } x<5 \end{cases}$

Proof: To prove that f is continuous to the right at $x=5$ i.e

$5<x<5+\delta$.

From the definition of f, $f(5)=25$. Let $\varepsilon>0$ be given then $\exists\,\delta \ni$

$|f(x)-f(5)| = |x^2-25| = |(x+5)(x-5)| = |x+5||x-5|$ but $|x-5|<\delta$. if $\delta<1$

then $|x-5|<1 \Rightarrow 5<x<6$

$\therefore |f(x)-f(5)| = |x-3||x+3| < |x+3|\delta < 11\delta$

So we choose $\delta = \min(1, \varepsilon/11)$

Therefore, f is continuous the right at $x=5$ ◻.

(5a) $f(x)=x^3-4x$ on the interval $[-2,2]$

We first check if $f(x)$ is continuous. By definition: let $\varepsilon>0$ be given, then we find $\delta>0 \ni |f(x)-f(x_0)|<\varepsilon$ whenever $|x-a|<\delta$; $a\in[-2,2]$

$|f(x)-f(a)| = |x^3-4x-x_0^3+4(x_0)| \leq |x^3-x_0^3| + |4(x_0-x)|$

$= |(x^2+x_0x+x_0^2)(x-x_0)| + 4|x_0-x| \leq |x^2+x_0x+x_0^2||x-x_0| + 4|x-x_0|$

$= |x-x_0|(x^2+x_0x+x_0^2+4) < \delta(x^2+x_0x+x_0^2+4)$

Since $x, x_0\in[-2,2)$ $|f(x)-f(x_0)|<\delta(4+4+4+4) = 16\delta = \varepsilon$

So, $\delta = \varepsilon/16$

Hence, f is continuous for all $x\in[-2,2]$

$f(a)=f(-2) = (-2)^3-4(-2) = -8+8=0$

$f(b) = f(2) = 2^3-4(2) = 8-8=0$

$f(2) = f(-2)$

Differentiable on $(-2,2)$, $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{x^3-4x-(x_0^3-4(x_0))}{x-x_0}$

$= \lim_{x\to x_0} \frac{x^3-x_0^3-4(x-x_0)}{x-x_0} = \lim_{x\to x_0} \frac{x^3-x_0^3-4(x-x_0)}{x-x_0} = \lim_{x\to x_0} \frac{(x^2+xx_0+x_0^2)(x-x_0)-4(x-x_0)}{x-x_0}$

$= \lim_{x\to x_0} x^2+xx_0+x_0^2-4 = x_0^2+x_0^2+x_0^2-4 = 3x_0^2-4$ $\exists\, c \in (-2,2) \ni f'(c) = 0$

i.e $f'(c) = 3c^2-4 = 0 \Rightarrow 3c^2 = 4 \Rightarrow c = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$

Hence, the value of c in $(-2,2)$ such that $f'(c)=0$

$\Rightarrow c = \frac{2}{\sqrt{3}}$ or $\frac{2\sqrt{3}}{3}$

(5b) $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

To show that f is differentiable at $x=0$ but $\lim_{x\to 0} f'(x) \neq f'(0)$

Differentiable at $x=0 \Rightarrow f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2\sin(1/h)}{h} = \lim_{h\to 0} h\sin(1/h)$

Hence, $f'(0) = 0$.

By computing $f'(x)$ for $x \neq 0$, Using product and chain rules;

$f'(x) = 2x\sin\frac{1}{x} + x^2\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$, $(x\neq 0)$

$\Rightarrow \lim_{x\to 0} f'(x)$ does not equal $f'(0)$

$2x\sin\left(\frac{1}{x}\right) \longrightarrow 0$ as $x \to 0$ but $-\cos\left(\frac{1}{x}\right)$ Oscillates.

$\lim_{x\to 0} f'(x)$ does not exist $\neq f'(0) = 0$

Hence, f is differentiable at 0 with $f'(0)=0$ but $\lim_{x\to 0} f'(x)$ does not exist

$\Rightarrow \lim_{x\to 0} f'(x) \neq f'(0)$ $\square$.

(6a) Proof: f is Uniformly Continuous, given $\varepsilon > 0$ $\exists\, \delta > 0$ such that $x_1, x_2 \in X$

$|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$.

By Contradiction, Suppose f is continuous but not Uniformly Continuous $\forall$

$x_1, x_2 \in [a,b]$. $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| \geq \varepsilon$.

Now, for each $\delta_n = \frac{1}{n}$, $\exists\, x_n, x'_n$ such that $|x_n - x'_n| < 1/n$ and $|f(x_n) - f(x'_n)| \geq \varepsilon$

But $\{x_n\} \subseteq [a,b]$, so $\{x_n\}$ is a bounded subsequence and by Bolzano Weierstrass

theorem, $\{x_n\}_{n=1}^{\infty}$ has a Convergent Subsequence, say $\{x_{n_k}\}_{k=1}$ i.e

$x_{n_k} \to x_0 \in [a,b]$ from $|x_n - x'_n| = 1/n$, we obtain $x_{n_k} \longrightarrow x_0$.

Since f is continuous at x_0, then $\exists\, \delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon/2$

Now, there is an integer N st $|x_n - x_0| < \delta$ and $|x'_{n_k} - x_0| < \delta$ then $\forall j \geq N$

$|f(x_{n_k}) - f(x'_{n_k})| < |f(x_{n_k}) - f(x_0)| + |f(x_0) - f(x'_{n_k})|$

$\leq |f(x_{n_k}) - f(x_0)| + |f(x'_{n_k}) - f(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Which Contradict to the fact that $|f(x_1) - f(x_2)| > \varepsilon$

Thus f is Uniformly Continuous on $[a,b]$.

$\square$.

6b) Proof: For any $x, y \in [-1,1]$

$|f(x) - f(y)| = |x^2 - y^2| = |x-y||x+y| \leq |x-y| \cdot 2$

$= 2|x-y|$

$\Rightarrow |f(x) - f(y)| \leq 2|x-y| < 2 \cdot (\varepsilon/2) = \varepsilon$.

Hence $f(x) = x^2$ is uniformly Continuous on the closed interval $[-1,1]$.

$\square$.